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RESEARCH ARTICLE

H. M. Barakat

The maximal correlation for the generalized order statistics and dual generalized order statistics

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Abstract In each of three exhaustive and distinct cases, it is found a distribution for which the correlation coefficient between the elements of the generalized order statistics (gos) is maximal. The corresponding result for the dual generalized order statistics (dgos) is derived for other three different distributions. Moreover, some interesting relations for the regression curves between the elements of gos and dgos based on these distributions are obtained. As a consequence of this result, a non-parametric criterion of independence between gos and between dgos in a general setting is derived.

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الملخص

في كل حالة من ثلاث حالات حصرية ومتميزة، وجد أن توزيع معامل الارتباط بين عناصر الإحصاءات المرتبة المعممة (gos) متعاظم. وقد تم اشتقاق نتيجة مناظرة للإحصاءات المرتبة المعممة البديلة (dgos) مقابلة لثلاثة توزيعات أخرى. وعلاوة على ذلك تم الحصول علي بعض علاقات هامة لمنحنيات الانحدار بين عناصر gos و dgos بالنسبة لتلك التوزيعات. وكان من نتائج تلك الدراسة اشتقاق معيار غير مَعلمي لاختبار الاستقلال بين عناصر gos و dgos وذلك في إطار عام.

1 Introduction

Kamps [9] introduced the concept of the generalized order statistics (gos) as a unification of several models of ascendingly ordered random variables (rvs). It is known that ordinary order statistics (oos), upper record values, sequential order statistics and progressive type II censored order statistics are special cases of gos. Uniform gos $U^\star(r) \equiv U(r, n, k, \tilde{m})$, $r = 1, 2, \ldots, n$, are defined by their density function

$$f^{U^\star(1),\ldots,U^\star(n)}(u_1, \ldots, u_n) = \left(\prod_{j=1}^{n} \gamma_j\right) \left(\prod_{j=1}^{n-1} (1 - u_j)^{\gamma_j - \gamma_{j+1} - 1}\right) (1 - u_n)^{\gamma_n - 1}$$

on the cone $\{(u_1, \ldots, u_n) : 0 \le u_1 \le \cdots \le u_n < 1\} \subset \Re^n$, with parameters $\gamma_1, \ldots, \gamma_n > 0$. The parameters $\gamma_1, \ldots, \gamma_n$ are defined by $\gamma_n = k > 0$ and $\gamma_r = k + n - r + M_r$, $r = 1, 2, \ldots, n - 1$, where $M_r = \sum_{j=r}^{n-1} m_j$ and $\tilde{m} = (m_1 m_2 \ldots m_{n-1}) \in \Re^{n-1}$. gos based on some distribution function (df) F are defined via the quantile

H. M. Barakat (✉)
Department of Mathematics, Faculty of Science, Zagazig University,
Zagazig, Egypt
E-mail: hbarakat2@hotmail.com

transformation $X^\star(r) = F^{-1}(U^\star(r)), r = 1, 2, \ldots, n$. Consequently, $X^\star(1) \le X^\star(2) \le \cdots \le X^\star(n)$ holds almost surely. Burkschat et al. [3] have introduced the concept of dual generalized order statistics (dgos) to enable a common approach to descendingly ordered rvs like reversed order statistics and lower record values. Uniform dgos $U_d^\star(r) \equiv U_d(r, n, k, \tilde{m}), r = 1, 2, \ldots, n$, are defined by their density function

$$f^{U_d^\star(1),\ldots,U_d^\star(n)}(u_1,\ldots,u_n) = \left(\prod_{j=1}^{n} \gamma_j\right)\left(\prod_{j=1}^{n-1} u_j^{\gamma_j-\gamma_{j+1}-1}\right) u_n^{\gamma_n-1}$$

on the cone $\{(u_1,\ldots,u_n) : 1 \ge u_1 \ge \cdots \ge u_n > 0\} \subset \Re^n$. The quantile transformation $X_d^\star(r) = F_d^{-1}(U_d^\star(r)), r = 1, 2, \ldots, n$, yields dgos based on arbitrary df F_d. Therefore, $X_d^\star(1), \ldots, X_d^\star(n)$ are arranged in descending order almost surely.

Let $B_j, 1 \le j \le n$, be independent rvs with respective Beta distribution $Beta(\gamma_j, 1)$, i.e., B_j follows a power function distribution with exponent γ_j. The central distribution theoretical result concerning gos and dgos is that they can, respectively, be defined by the product of the independent power function distributed rvs $B_j, 1 \le j \le n$ (see Cramer [5] and Burkschat et al. [3]). Namely,

$$X^\star(r) \sim F^{-1}\left(1 - \prod_{j=1}^{r} B_j\right), \quad r = 1, 2, \ldots, n, \tag{1.1}$$

and

$$X_d^\star(r) \sim F_d^{-1}\left(\prod_{j=1}^{r} B_j\right), \quad r = 1, 2, \ldots, n. \tag{1.2}$$

In the classical oos, where $\tilde{m} = (00\cdots 0)$ and $k = 1$, Terrell [15] proved that, when the sample size $n = 2$, and if the oos $X_{1:2}$ and $X_{2:2} (X_{1:2} \le X_{2:2})$ have finite variances, then their correlation coefficient satisfies the inequality $\mathrm{corr}(X_{1:2}, X_{2:2}) \le 1/2$ with equality if, and only if, F is uniformly distributed. Later, by considering the maximum correlation possible between any square-integrable functions of the uniform oos, and using the modified Jacobi polynomial, Székely and Móri [14] have shown that

$$\mathrm{corr}(X_{r:n}, X_{s:n}) \le \sqrt{\frac{r(n-s+1)}{s(n-r+1)}}, \quad 1 \le r < s \le n, \tag{1.3}$$

(here it is supposed that $\mathrm{var}(X_{r:n})$ and $\mathrm{var}(X_{s:n})$ are finite and the sample size can be arbitrary). An interesting alternative proof of (1.3) is given by Rohatgi and Székely [12] (see David and Nagaraja [6]). In this paper, using the method of Rohatgi and Székely [12], we extend the above result to a wide subclass of gos and dgos. Namely, for any $1 \le r < s \le n$, we consider the gos $X^\star(r), X^\star(s)$ and the dgos $X_d^\star(r), X_d^\star(s)$ for which $m_1 = m_2 = \cdots m_{s-1} = m$. Moreover, we consider the three exhaustive and distinct cases $m+1 > 0$, $m+1 = 0$ and $m + 1 < 0$. Clearly the $m-$gos and $m-$dgos, where $m_1 = m_2 = \cdots = m_{n-1} = m$ (see Kamps [9] and Burkschat et al. [3]), are special cases of these subclasses. Using the results of Kamps [9], Cramer [5] and Burkschat et al. [3], we can write explicitly the conditional density functions of $X^\star(r)$ given $X^\star(s)$, $X^\star(s)$ given $X^\star(r)$, $X_d^\star(r)$ given $X_d^\star(s)$ and $X_d^\star(s)$ given $X_d^\star(r)$, respectively, as:

$$\begin{aligned} f^{r|s}(x_r|x_s) &= f^{X^\star(r)|X^\star(s)}(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!}(1-F(x_r))^m g_m^{r-1}(F(x_r)) g_m^{-s+1}(F(x_s)) \\ &\quad [h_m(F(x_s)) - h_m(F(x_r))]^{s-r-1} f(x_r),\ F^{-1}(0) < x_r < x_s < F^{-1}(1), \end{aligned} \tag{1.4}$$

$$\begin{aligned} f^{s|r}(x_s|x_r) &= f^{X^\star(s)|X^\star(r)}(x_s|x_r) = \frac{C_{s-1}}{C_{r-1}(s-r-1)!}(1-F(x_r))^{-\gamma_r+m+1}(1-F(x_s))^{\gamma_s-1} \\ &\quad [h_m(F(x_s)) - h_m(F(x_r))]^{s-r-1} f(x_s),\ F^{-1}(0) < x_r < x_s < F^{-1}(1), \end{aligned} \tag{1.5}$$

$$\begin{aligned} f_d^{r|s}(x_r|x_s) &= f^{X_d^\star(r)|X_d^\star(s)}(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!} F_d^m(x_r) g_m^{r-1}(1-F_d(x_r)) \\ &\quad g_m^{-s+1}(1-F_d(x_s))[h_m(1-F_d(x_s)) - h_m(1-F_d(x_r))]^{s-r-1} f_d(x_r),\ F_d^{-1}(0) < x_s < x_r < F_d^{-1}(1), \end{aligned} \tag{1.6}$$

and

$$f_d^{s|r}(x_s|x_r) = f^{X_d^\star(s)|X_d^\star(r)}(x_s|x_r) = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} F_d^{-\gamma_r+m+1}(x_r) F_d^{\gamma_s-1}(x_s)$$
$$[h_m(1-F_d(x_s)) - h_m(1-F_d(x_r))]^{s-r-1} f_d(x_s),\ F_d^{-1}(0) < x_s < x_r < F_d^{-1}(1), \tag{1.7}$$

where

$$g_m(x) = \begin{cases} \frac{1}{m+1}[1-(1-x)^{m+1}], & m \neq -1, \\ -\log(1-x), & m = -1, \end{cases}$$
$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1, \\ -\log(1-x), & m = -1, \end{cases}$$

$C_{r-1} = \prod_{i=1}^{r} \gamma_r$, $r = 1, 2, \ldots, n$, with $\gamma_n = k$, $f(x) = \frac{\partial F(x)}{\partial x}$ and $f_d(x) = \frac{\partial F_d(x)}{\partial x}$.

Remark 1.1 For all $1 \le r < s \le n$ and all values of m, it is easy to prove the following relations

(1) $\gamma_s - \gamma_r = -(s-r)(m+1)$

(2) $\frac{C_{s-1}}{C_{r-1}} = \prod\limits_{i=r+1}^{s} \gamma_i = \begin{cases} (m+1)^{s-r} \prod_{j=1}^{s-r}(\frac{\gamma_s}{m+1} + s - r - j), & m \neq -1, \\ \gamma_s^{s-r}, & m = -1. \end{cases}$

2 The main result

Theorem 2.1 *Let $X^\star(1) \le X^\star(2) \le \cdots \le X^\star(n)$ and $X_d^\star(n) \le X_d^\star(n-1) \le \cdots \le X_d^\star(1)$ be gos and dgos based on arbitrary continuous df's F and F_d, respectively, such that for any $1 \le r < s \le n$ we have $m_1 = m_2 = \cdots = m_{s-1} = m$. Furthermore, let $X^\star(r)$, $X^\star(s)$, $X_d^\star(r)$ and $X_d^\star(s)$ have finite variances. Then*

$$corr(X^\star(r), X^\star(s)) \le \rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{A_s(m)}{A_r(m)} \times \frac{B_r(m) - A_r(m)}{B_s(m) - A_s(m)}}, \tag{2.1}$$

and

$$corr(X_d^\star(r), X_d^\star(s)) \le \rho_n(r, s, m, \tilde{\gamma}_s), \tag{2.2}$$

where

$$A_l(m) = \begin{cases} \prod_{j=1}^{l} \frac{\beta_j}{1+\beta_j}, & m \neq -1, \\ 1, & m = -1, \end{cases} \tag{2.3}$$

$$B_l(m) = \begin{cases} \prod_{j=1}^{l} \frac{\beta_j+1}{\beta_j+2}, & m \neq -1, \\ 1 + \sum_{j=1}^{l} \frac{1}{\gamma_j^2}, & m = -1, \end{cases} \tag{2.4}$$

$\beta_j = \frac{\gamma_j}{m+1}$, with $\beta_1 = \frac{\gamma_1}{m+1} < -2$, if $m+1 < 0$, and $\tilde{\gamma}_s = (\gamma_1\gamma_2\ldots\gamma_s)$. Moreover (2.1) and (2.2) are satisfied with equality if

$$F(x) = \begin{cases} F_1(x) = 1 - (1-x)^{\frac{1}{m+1}}, 0 < x < 1, & \text{if } m+1 > 0, \\ F_2(x) = 1 - e^{-x}, 0 < x < \infty, & \text{if } m = -1, \\ F_3(x) = 1 - x^{\frac{1}{m+1}}, 1 < x < \infty, & \text{if } m+1 < 0, \end{cases} \tag{2.5}$$

and

$$F_d(x) = \begin{cases} F_{1d}(x) = x^{\frac{1}{m+1}}, 0 < x < 1, & \text{if } m+1 > 0, \\ F_{2d}(x) = e^{x}, -\infty < x < 0, & \text{if } m = -1, \\ F_{3d}(x) = (1-x)^{\frac{1}{m+1}}, -\infty < x < 0, & \text{if } m+1 < 0. \end{cases} \tag{2.6}$$

Corollary 2.1 *Let* $\mu_i^{r|s}(x_s) = E(X_i^\star(r)|X_i^\star(s) = x_s)$ *be the regression curve of* $X_i^\star(r)$ *given* $X_i^\star(s)$*, where* $X_i^\star(r)$ *and* $X_i^\star(s)$ *are the gos based on the df* $F_i(x)$, $i = 1, 2, 3$. *Similarly, let* $\mu_{id}^{r|s}(x_s) = E(X_{id}^\star(r)|X_{id}^\star(s) = x_s)$ *be the regression curve of* $X_{id}^\star(r)$ *given* $X_{id}^\star(s)$*, where* $X_{id}^\star(r)$ *and* $X_{id}^\star(s)$ *are the dgos based on the df* $F_{id}(x)$, $i = 1, 2, 3$. *Then, we have the following relations:*

(1) $\mu_{1d}^{r|s}(x_s) - \mu_1^{r|s}(x_s) = 1 - \frac{r}{s}$, *for all* $0 < x_s < 1, m + 1 > 0$.
(2) $\mu_1^{s|r}(x_r) - \mu_{1d}^{s|r}(x_r) = \frac{(s-r)(m+1)}{\gamma_s+(s-r)(m+1)}$, *for all* $0 < x_r < 1, m + 1 > 0$.
(3) $\mu_2^{r|s}(x_s) + \mu_{2d}^{r|s}(-x_s) = 0$, *for all* $0 < x_s < \infty, m = -1$.
(4) $\mu_2^{s|r}(x_r) + \mu_{2d}^{s|r}(-x_r) = 0$, *for all* $0 < x_r < \infty, m = -1$.
(5) $\mu_3^{r|s}(x_s) + \mu_{3d}^{r|s}(-x_s) = 1 - \frac{r}{s}$, *for all* $1 < x_s < \infty, m + 1 < 0$.
(6) $\mu_3^{s|r}(x_r) + \mu_{3d}^{s|r}(-x_r) = \frac{(s-r)(m+1)}{\gamma_s+(s-r)(m+1)}$, *for all* $1 < x_r < \infty, m + 1 < 0$.

Proof of Theorem 2.1 Following the method of Rohatgi and Székely [12], we see that the proof of Theorem 2.1 will depend on the Sarmanov [13] result, which states that if X and Y are arbitrary rvs with finite variances and $\mathrm{E}(X|Y)$ and $\mathrm{E}(Y|X)$ are both linear, then for any measurable functions ϕ and ψ, for which $\mathrm{var}(\phi(X))$ and $\mathrm{var}(\phi(Y))$ are finite

$$\sup_{\phi,\psi} \mathrm{corr}(\phi(X), \psi(Y)) = |\mathrm{corr}(X, Y)|. \tag{2.7}$$

Therefore, the first step of our proof is to check the regression curves $\mu_i^{r|s}(x_s), \mu_i^{s|r}(x_r), \mu_{id}^{r|s}(x_s)$ and $\mu_{id}^{s|r}(x_r), i = 1, 2, 3$. On the other hand, we have $F_t(X_t^\star(r)) = U^\star(r), t = 1, 2, 3$, where $U^\star(1) \leq \cdots \leq U^\star(r) \leq \cdots \leq U^\star(n)$ are the uniform gos. Thus, for any $m-$gos $X^\star(r)$ based on an arbitrary continuous df F, we get $X^\star(r) = F^{-1}(U^\star(r)) = F^{-1}(F_t(X_t^\star(r)))$, where $t = 1, 2, 3$ if $m > -1, m = -1, m < -1$, respectively. This means that, for every df F, there exists a function $\phi_t(x)$ such that $X^\star(r) = \phi_t(X_t^\star(r))$ and $X^\star(s) = \phi_t(X_t^\star(s)), t = 1, 2, 3$ (e.g., $\phi_1(x) = F^{-1}(F_1(x)) = F^{-1}(1 - (1 - x)^{\frac{1}{m+1}})$ and $\phi_2(x) = F^{-1}(F_2(x)) = F^{-1}(1 - e^{-x})$. Moreover, for the case of the oos and the upper record values which are considered in Rohatgi and Székely [12], we have $\phi_1(x) = F^{-1}(x)$ and $\phi_2(x) = F^{-1}(1 - e^{-x})$, with $X_1^\star(i) = U_{i:n}$ is the ith uniform oos and $X_2^\star(i) = R_i$ is the ith upper record value based on the exponential distribution). Thus (2.7) implies that

$$\mathrm{corr}(X^\star(r), X^\star(s)) \leq \mathrm{corr}(X_t^\star(r), X_t^\star(s)),$$

where $t = 1$ if $m + 1 > 0$, $t = 2$ if $m + 1 = 0$ and $t = 3$ if $m + 1 < 0$. A similar argument proves

$$\mathrm{corr}(X_d^\star(r), X_d^\star(s)) \leq \mathrm{corr}(X_{td}^\star(r), X_{td}^\star(s)),$$

where $t = 1$ if $m + 1 > 0$, $t = 2$ if $m + 1 = 0$ and $t = 3$ if $m + 1 < 0$.

Now, if $m + 1 > 0$, one can easily verify that

$$\mu_1^{r|s}(x_s) = \int_0^{x_s} x_r f^{r|s}(x_r|x_s)dx_r = \frac{r}{s}x_s, \tag{2.8}$$

using (1.4) with F_1 and the transformation $x_r = x_s z$,

$$\mu_1^{s|r}(x_r) = 1 - \int_{x_r}^1 (1 - x_s) f^{s|r}(x_s|x_r)dx_s = \frac{\gamma_s}{\gamma_s + (s-r)(m+1)}x_r + \frac{(s-r)(m+1)}{\gamma_s + (s-r)(m+1)}, \tag{2.9}$$

using (1.5) with F_1, the transformation $1 - x_s = (1 - x_r)z$ and Remark (1.1),

$$\mu_{1d}^{r|s}(x_s) = \int_{x_s}^1 x_r f_d^{r|s}(x_r|x_s)dx_r = \frac{r}{s}x_s + \left(1 - \frac{r}{s}\right), \tag{2.10}$$

using (1.6) with F_{1d} and the transformation $x_r = x_s + (1 - x_s)z$, and

$$\mu_{1d}^{s|r}(x_r) = \int_0^{x_r} x_s f_d^{s|r}(x_s|x_r)dx_s = \frac{\gamma_s}{\gamma_s + (s-r)(m+1)} x_r, \tag{2.11}$$

using (1.7) with F_{1d}, the transformation $x_s = x_r z$ and Remark (1.1). Similarly, if $m = -1$, we get

$$\mu_2^{r|s}(x_s) = \int_0^{x_s} x_r f^{r|s}(x_r|x_s)dx_r = \frac{r}{s} x_s, \tag{2.12}$$

using (1.4) with F_2 and the transformation $x_r = x_s z$,

$$\mu_2^{s|r}(x_r) = x_r + \int_{x_r}^{\infty} (x_s - x_r) f^{s|r}(x_s|x_r)dx_s = x_r + \frac{s-r}{\gamma_s}, \tag{2.13}$$

using (1.5) with F_2, the transformation $x_s = x_r + z$ and Remark (1.1),

$$\mu_{2d}^{r|s}(x_s) = \int_{x_s}^{0} x_r f_d^{r|s}(x_r|x_s)dx_r = \frac{r}{s} x_s, \tag{2.14}$$

using (1.6) with F_{2d} and the transformation $x_r = x_s z$, and

$$\mu_{2d}^{s|r}(x_r) = \int_{-\infty}^{x_r} x_s f_d^{s|r}(x_s|x_r)dx_s = x_r - \frac{s-r}{\gamma_s}, \tag{2.15}$$

using (1.7) with F_{2d}, the transformation $x_s = x_r - z$ and Remark (1.1). Finally, if $m + 1 < 0$, one can easily check that

$$\mu_3^{r|s}(x_s) = \int_1^{x_s} x_r f^{r|s}(x_r|x_s)dx_r = \frac{r}{s} x_s + \left(1 - \frac{r}{s}\right), \tag{2.16}$$

using (1.4) with F_3 and the transformation $x_r = x_s + (1 - x_s)z$,

$$\mu_3^{s|r}(x_r) = \int_{x_r}^{\infty} x_s f^{s|r}(x_s|x_r)dx_s = \frac{\gamma_s}{\gamma_s + (s-r)(m+1)} x_r, \tag{2.17}$$

using (1.5) with F_3, the transformation $x_s = \frac{x_r}{z}$ and Remark (1.1),

$$\mu_{3d}^{r|s}(x_s) = \int_{x_s}^{0} x_r f_d^{r|s}(x_r|x_s)dx_r = \frac{r}{s} x_s, \tag{2.18}$$

using (1.6) with F_{3d} and the transformation $x_r = x_s z$, and

$$\mu_{3d}^{s|r}(x_r) = 1 - \int_{-\infty}^{x_r} (1 - x_s) f_d^{s|r}(x_s|x_r)dx_s = \frac{\gamma_s}{\gamma_s + (s-r)(m+1)} x_r + \frac{(s-r)(m+1)}{\gamma_s + (s-r)(m+1)}, \tag{2.19}$$

using (1.7) with F_{3d}, the transformation $1 - x_s = \frac{1-x_r}{1-z}$ and Remark (1.1).

The relations (2.8)–(2.19) show that all the regression curves $\mu_i^{r|s}(x_s)$, $\mu_i^{s|r}(x_r)$, $\mu_{id}^{r|s}(x_s)$, and $\mu_{id}^{s|r}(x_r)$, $i = 1, 2, 3$, are linear. Finally, the following lemma completes the proof of Theorem 2.1.

Lemma 2.1 *For any* $1 \le r < s \le n$, *we have*

$$corr(X_t^\star(r), X_t^\star(s)) = corr(X_{td}^\star(r), X_{td}^\star(s)) = \sqrt{\frac{A_s(m)}{A_r(m)} \times \frac{B_r(m) - A_r(m)}{B_s(m) - A_s(m)}} = \rho_n(r, s, m, \tilde{\gamma}_s), \quad (2.20)$$

where $A_l(m)$ *and* $B_l(m)$ *are defined in* (2.3) *and* (2.4), *respectively.*

Proof We first consider the case $t = 1$, i.e., $F_1(x)$, $F_{1d}(x)$. In this case (1.1) and (1.2) take, respectively, the forms $X_1^\star(r) \sim 1 - \prod_{j=1}^r D_j$ and $X_{1d}^\star(r) \sim \prod_{j=1}^r D_j$, where $D_j = B_j^{m+1}$, $1 \le j \le n$, be independent rvs with respective power function distribution with exponent $\beta_j = \frac{\gamma_j}{m+1}$. Therefore, $\mathrm{E}(D_j^k) = \frac{\beta_j}{\beta_j + k}$, $k = 1, 2, \ldots$, and $\mathrm{E}(X_1^\star(r)) = 1 - A_r(m)$, $\mathrm{E}(X_{1d}^\star(r)) = A_r(m)$. Also, it can be easily shown that $\mathrm{E}(X_1^\star(r) X_1^\star(s)) = 1 - A_r(m) - A_s(m) + \prod_{j=1}^r \frac{\beta_j}{2+\beta_j} \prod_{j=r+1}^s \frac{\beta_j}{1+\beta_j}$ and $\mathrm{E}(X_{1d}^\star(r) X_{1d}^\star(s)) = \prod_{j=1}^r \frac{\beta_j}{2+\beta_j} \prod_{j=r+1}^s \frac{\beta_j}{1+\beta_j}$. Thus, we get

$$\begin{aligned}
\mathrm{cov}(X_1^\star(r), X_1^\star(s)) &= \mathrm{cov}(X_{1d}^\star(r), X_{1d}^\star(s)) \\
&= \prod_{j=1}^r \frac{\beta_j}{2+\beta_j} \prod_{j=r+1}^s \frac{\beta_j}{1+\beta_j} - A_r(m) A_s(m) \\
&= \prod_{j=r+1}^s \frac{\beta_j}{1+\beta_j} \left[\prod_{j=1}^r \frac{\beta_j}{2+\beta_j} - \prod_{j=1}^r \left(\frac{\beta_j}{1+\beta_j} \right)^2 \right] \\
&= \prod_{j=1}^s \frac{\beta_j}{1+\beta_j} \left[\prod_{j=1}^r \frac{1+\beta_j}{2+\beta_j} - \prod_{j=1}^r \frac{\beta_j}{1+\beta_j} \right] = A_s(m)(B_r(m) - A_r(m)). \quad (2.21)
\end{aligned}$$

On the other hand, using (2.21), we get

$$\mathrm{var}(X_1^\star(r)) = \mathrm{var}(X_{1d}^\star(r)) = A_r(m)(B_r(m) - A_r(m)). \quad (2.22)$$

By combining (2.21) with (2.22), we get the desired relation (2.20), in this case.
Second, we consider the case $F_2(x)$, $F_{2d}(x)$. In this case (1.1) and (1.2) take, respectively, the forms $X_2^\star(r) \sim -\sum_{j=1}^r \log B_j$ and $X_{2d}^\star(r) \sim \sum_{j=1}^r \log B_j$, where B_j be independent rvs with respective Beta distribution $Beta(\gamma_j, 1)$. Therefore, $\mathrm{E}((-\log B_j)^k) = \frac{\Gamma(k+1)}{\gamma_j^k}$, $k = 1, 2, \ldots$, and thus $\mathrm{E}(X_2^\star(r)) = \sum_{j=1}^r \frac{1}{\gamma_j}$ and $\mathrm{var}(X_2^\star(r)) = \sum_{j=1}^r \frac{1}{\gamma_j^2}$. Moreover, we have

$$\begin{aligned}
\mathrm{E}(X_2^\star(r) X_2^\star(s)) &= \mathrm{E}\left(\sum_{j=1}^r \left(\log \frac{1}{B_j} \right)^2 + \sum_{i=1_{i\neq j}}^r \sum_{j=1}^r \left(\log \frac{1}{B_i} \right) \left(\log \frac{1}{B_j} \right) + \sum_{i=1}^r \sum_{j=r+1}^s \left(\log \frac{1}{B_i} \right) \left(\log \frac{1}{B_j} \right) \right) \\
&= \sum_{j=1}^r \frac{2}{\gamma_j^2} + \sum_{i=1_{i\neq j}}^r \sum_{j=1}^r \frac{1}{\gamma_i \gamma_j} + \sum_{i=1}^r \sum_{j=r+1}^s \frac{1}{\gamma_i \gamma_j}.
\end{aligned}$$

Therefore, we get

$$\mathrm{cov}(X_2^\star(r), X_2^\star(s)) = \sum_{j=1}^r \frac{2}{\gamma_j^2} + \sum_{i=1_{i\neq j}}^r \sum_{j=1}^r \frac{1}{\gamma_i \gamma_j} + \sum_{i=1}^r \sum_{j=r+1}^s \frac{1}{\gamma_i \gamma_j} - \sum_{i=1}^r \sum_{j=1}^s \frac{1}{\gamma_i \gamma_j} = \sum_{j=1}^r \frac{1}{\gamma_j^2},$$

and, thus

$$\mathrm{corr}(X_2^\star(r), X_2^\star(s)) = \sqrt{\frac{\sum_{j=1}^r \frac{1}{\gamma_j^2}}{\sum_{j=1}^s \frac{1}{\gamma_j^2}}} = \sqrt{\frac{A_s(-1)}{A_r(-1)} \times \frac{B_r(-1) - A_r(-1)}{B_s(-1) - A_s(-1)}}.$$

On the other hand, since $X^{\star}_{2d}(r) = -X^{\star}_2(r)$, we get $\mathrm{corr}(X^{\star}_{2d}(r), X^{\star}_{2d}(s)) = \mathrm{corr}(-X^{\star}_2(r), -X^{\star}_2(s)) = \mathrm{corr}(X^{\star}_2(r), X^{\star}_2(s))$. This completes the proof of (2.20), in the second case.

Finally, we consider the case $F_3(x), F_{3d}(x)$. In this case (1.1) and (1.2) take, respectively, the forms $X^{\star}_3(r) \sim \prod_{j=1}^{r} D_j$ and $X_{3d}(r) \sim 1 - \prod_{j=1}^{r} D_j$, where $D_j = B_j^{m+1}, 1 \le j \le n$. Therefore, $\mathrm{E}(D_j^k) = \frac{\beta_j}{\beta_j + k}, k = 1, 2, \ldots,$ provided that $\beta_j < -k$. Thus, the kth moments of $X^{\star}_3(l)$ and $X^{\star}_{3d}(l), l = r, s$, exist provided that $\beta_j < -k, j = 1, 2, \ldots, s$, or equivalently $\gamma_j + k(m+1) > 0, j = 1, 2, \ldots, s$. On the other hand, in view of Remark (1.1), we have $\gamma_{j+1} + 2(m+1) = \gamma_j + (m+1), j = 1, 3, \ldots, s-1$. The last relation with the inequality $\gamma_j + (m+1) > \gamma_j + 2(m+1), j = 1, 2, \ldots, s$, show that the condition $\beta_1 < -2$ (or equivalently $\gamma_1 + 2(m+1) > 0$), implies there exist first and second moments of $X^{\star}_3(l)$ and $X^{\star}_{3d}(l), l = r, s$. Now, we can easily see that the proof of (2.20) in this case is similar as the proof of the first case. The proof of Lemma 2.1, as well as the proof of Theorem 2.1. are completed. □

Proof of Corollary 2.1 The proof immediately follows using the relations (2.8)–(2.19).

Remark 2.1 The maximal coefficient of correlation between a pair of rvs (X, Y), introduced by [8], is defined by the left hand side of equation (2.7). Therefore,

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{A_s(m)}{A_r(m)} \times \frac{B_r(m) - A_r(m)}{B_s(m) - A_s(m)}}$$

is the maximal coefficient of correlation between a pair of gos $(X^{\star}(r), X^{\star}(s))$ and a pair of dgos $(X^{\star}_d(r), X^{\star}_d(s))$, based on arbitrary continuous df's F andF_d, respectively. Rényi [11] gives a set of seven postulates which a measure of dependence for a pair of rvs should satisfy. Of the dependence measures considered by Rényi, only the maximal coefficient of correlation satisfies all seven postulates. Consequently, the maximal coefficient of correlation is conveniently applied to problems whose solution is considerably determined by characteristics of stochastic dependence such as the statistical linearization (e.g., Chernyshov [4]). The maximal coefficient of correlation, besides being a convenient measure of dependence, plays a critical role in various areas of statistics including correspondence analysis, optimal transformation for regression, and the theory of Markov processes, see Yaming [16]. Finally, It is worth remarking that in Theorem 2.1 the maximal correlation coefficient between pairs of rvs is explicitly computed. Such exact computations are relatively rare. One well-known case, the Gaussian case, is due to Lancaster [10]; another example of such an explicit computation is the case of partial sums of i.i.d. rvs considered by Dembo et al. [7]. The third such case is order statistics given by (1.3).

Remark 2.2 In the classical case of oos, where $m = 0$ and $\gamma_i = n - i + 1, 1 \le i \le n$, it is easy to show that

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{r(n-s+1)}{s(n-r+1)}}.$$

Moreover, $F_1(x) = F_{1d}(x) = x, 0 < x < 1$. This case is considered in Rohatgi and Székely [12] (the relation (1.3)), with $\mathrm{corr}(X_{r:n}, X_{s:n}) = \mathrm{corr}(X_{n-r+1:n}, X_{n-s+1:n})$, where $X^{\star}(r) = X_{r:n}, X^{\star}(s) = X_{s:n}, X^{\star}_d(r) = X_{n-r+1:n}$, and $X^{\star}_d(s) = X_{n-s+1:n}$). Finally, in the case of the upper record values, where $m = -1$ and $\gamma_i = 1, 1 \le i \le n$, it is easy to show that $\rho_n(r, s, -1, 1) = \sqrt{\frac{r}{s}}$. Moreover, $F_2(x) = 1 - e^{-x}, 0 < x < \infty$, which again leads to the result of Rohatgi and Székely [12] for the upper record values case (see relation 2 of Rohatgi and Székely [12]). For lower record values the above result holds with $F_{2d}(x) = e^x, -\infty < x < 0$. This result reflects the fact that the correlation coefficient between the elements of the upper records is maximal for the exponentially distributed populations, while the correlation coefficient between the elements of the lower records is maximal for the reflected exponential distribution $F_{2d}(x)$. Moreover, the maximal correlation coefficient between the elements of the upper records is the same as the maximal correlation coefficient between the elements of the lower records.

Remark 2.3 In Barakat [1] it is proved that $\rho_n(r, s, 0, \tilde{\gamma}_s) = \sqrt{\frac{A_s(0)}{A_r(0)} \times \frac{B_r(0) - A_r(0)}{B_s(0) - A_s(0)}}$ is the correlation coefficient between any two uniform gos $U^{\star}(r)$ and $U^{\star}(s)$ or any two uniform dgos $U^{\star}_d(r)$ and $U^{\star}_d(s)$, where no any restriction is imposed on the parameters $k, m_1, m_2, \ldots m_{n-1}$. Moreover, in the same paper, it is proved that the measure $\sigma^{\star}_{r,s:n} = 12\rho_n(r, s, 0, \tilde{\gamma}_s)$ provides a non-parametric criterion of asymptotically independence between the elements of gos and between the elements of dgos in general setting (where no any restriction is imposed on the parameters $k, m_1, m_2, \ldots m_{n-1}$). In the case of the upper and the lower record values we have $\sigma^{\star}_{r,s:n} = 12\sqrt{(\frac{3}{4})^{s-r} \times (\frac{1-(\frac{3}{4})^r}{1-(\frac{3}{4})^s})}$ (see Barakat [1]). In other hand, in view of Theorem 2.1, we get $\sigma^{\star}_{r,s:n} = 12\sqrt{(\frac{3}{4})^{s-r} \times (\frac{1-(\frac{3}{4})^r}{1-(\frac{3}{4})^s})} \leq 12\sqrt{\frac{r}{s}}$. The last relation reflects the fact that the asymptotic independence between any two oos $X_{r:n}$ and $X_{s:n}$ (which occurs, in view of the result of Barakat [1], if, and only if, $\frac{r}{s} \to 0$, as $n \to \infty$) implies the asymptotic independence between the upper records $R^{\star}_r$ and $R^{\star}_s$, as well as the lower records L_s and L_r (the asymptotic independence between the upper records $R^{\star}_r$ and $R^{\star}_s$, as well as the lower records occurs, in view of the result of Barakat [1], if, and only if, $s - r \to \infty$, as $n \to \infty$).

Remark 2.4 Since (2.20), with (2.3) and (2.4), is proved using the relations (1.1) and (1.2), we deduce that $\rho_n(r, s, m, \tilde{\gamma}_s)$ is the correlation coefficient between any two gos $X^{\star}_t(r)$ and $X^{\star}_t(s)$ based on the df F_t, $t = 1, 2, 3$, or between any two dgos $X^{\star}_{td}(r)$ and $X^{\star}_{td}(s)$ based on the df F_{td}, $t = 1, 2, 3$, where no any restriction imposed on the parameters $k, m_1, m_2, \ldots m_{n-1}$. Consequently, in this case the parameter m in $\rho_n(r, s, m, \tilde{\gamma}_s)$ is related to the df's F_t and F_{td}, $t = 1, 2, 3$, and not to the gos or dgos themselves.

3 Discussion and applications

The following two results are direct consequences of Theorem 2.1.

Corollary 3.1 *For any $r < s$, we have*

$$\rho_n(r, s, m, \tilde{\gamma}_s) \leq \min\left(\sqrt{\frac{A_s(m)}{A_r(m)}}, \sqrt{\frac{Br(m) - A_r(m)}{B_s(m) - A_s(m)}}\right). \tag{3.1}$$

Moreover, the asymptotic independence between the gos $X^{\star}_{r:n}$ and $X^{\star}_{s:n}$ occurs if, and only if, at least one of the relations $\frac{A_s(m)}{A_r(m)} \to 0$ and $\frac{B_r(m)-A_r(m)}{B_s(m)-A_s(m)} \to 0$ holds.

Proof Since $1 > \frac{1+\beta_j}{2+\beta_j} \geq \frac{\beta_j}{1+\beta_j} > 0$, for all j, we get $1 > B_k(m) \geq A_k(m) > 0$, for all k, and $1 > B_{kl}(m) \geq A_{kl}(m) > 0$, for all $k < l$, where $A_{kl}(m) = \prod_{j=k+1}^{l} \frac{\beta_j}{1+\beta_j}$ and $B_{kl}(m) = \prod_{j=k+1}^{l} \frac{1+\beta_j}{2+\beta_j}$. The combination of the preceding relations yields $0 < \frac{B_r(m)-A_r(m)}{B_s(m)-A_s(m)} = \frac{B_r(m)-A_r(m)}{B_{rs}(m)B_r(m)-A_{rs}(m)A_r(m)} < 1$, which completes the proof of (3.1), as well as the proof of Corollary 3.1. □

Corollary 3.2 *For any $r \neq s$ and any n, we have $X^{\star}_{r:n}$ and $X^{\star}_{s:n}$ are independent if, and only if, $X^{\star}_{d;r:n}$ and $X^{\star}_{d;s:n}$ are independent.*

Theorem 2.1 shows that for the dfs given in (2.5) (as well as (2.6)) are the maximal correlation between the elements of the gos (as well as the elements of dgos) equals the (Pearson) correlation. Thus, the uncorrelatedness of these elements implies their independence. This fact implies that in any real-world problems the linear relationship is the only possible relation between these elements. For example, the linear relationship is the only possible relation between any two upper records and between any two lower records for the exponentially and reflected exponentially distributed populations, respectively. Moreover, Theorem 2.1 enables us to derive some interesting results concerning the rates of the convergence to the asymptotic independence between different types of gos as well as dgos. The following consequence gives some of these results for the oos. Although, the proof of this consequence is simple, but to the best of the author knowledge this result is new.

Corollary 3.3 *Let $\underline{X}^E_{r_1:n}$, $\underline{X}^I_{r_2:n}$, $X^C_{r_3:n}$, $\overline{X}^I_{r_4:n}$ and $\overline{X}^E_{r_5:n}$ be the r_1th lower extreme (where r_1=constant, w.r.t. n), the r_2th lower intermediate (where $r_2 \to \infty$, $\frac{r_2}{n} \to 0$, as $n \to \infty$), the r_3th central (where $r_3 \to \infty$, $\frac{r_3}{n} \to \lambda \in (0, 1)$, as $n \to \infty$), the r_4th upper intermediate (where $r_4 \to \infty$, $\frac{r_4}{n} \to 1$, as $n \to \infty$) and the r_5th upper extreme (where $n - s_5$=constant, w.r.t. n) order statistic, respectively. Then*

(I) *the convergence to the asymptotic independence of the couple $(\underline{X}^E_{r_1:n}, \overline{X}^E_{r_5:n})$ is faster than the couple $(\underline{X}^I_{r_2:n}, \overline{X}^I_{r_4:n})$.*

(II) *the convergence to the asymptotic independence of the couple* $(\underline{X}^E_{r_1:n}, X^C_{r_3:n})$ *is faster than the couples* $(\underline{X}^E_{r_1:n}, \underline{X}^I_{r_2:n})$ *and* $(\underline{X}^I_{r_2:n}, X^C_{r_3:n})$.

(III) *the convergence to the asymptotic independence of the couple* $(\underline{X}^I_{r_2:n}, X^C_{r_3:n})$ *is faster than the couple* $(\underline{X}^E_{r_1:n}, \underline{X}^I_{r_2:n})$, *if and only if* $r_2 = \circ(\sqrt{n})$, *as* $n \to \infty$.

Proof The proof of (I), (II) and (III) follows immediately by noting, respectively, that

(I) $\rho_n(r_1, r_5, 0, \tilde{\gamma}_{r_5}), \rho_n(r_2, r_4, 0, \tilde{\gamma}_{r_4}) \to 0$ and $\dfrac{\rho_n(r_1, r_5, 0, \tilde{\gamma}_{r_5})}{\rho_n(r_2, r_4, 0, \tilde{\gamma}_{r_4})} \sim \sqrt{\dfrac{r_1(n - r_5 + 1)}{r_2(n - r_4 + 1)}} \to 0$, as $n \to \infty$,

since $r_1, n - r_5 + 1 =$ constant w.r.t. n and $r_2, n - r_4 + 1 \to \infty$, as $n \to \infty$,

(II) $\rho_n(r_1, r_2, 0, \tilde{\gamma}_{r_2}), \rho_n(r_1, r_3, 0, \tilde{\gamma}_{r_3}), \rho_n(r_2, r_3, 0, \tilde{\gamma}_{r_3}) \to 0$,

$$\frac{\rho_n(r_1, r_3, 0, \tilde{\gamma}_{r_3})}{\rho_n(r_1, r_2, 0, \tilde{\gamma}_{r_2})} \sim \sqrt{\frac{1-\lambda}{\lambda} \times \frac{r_2}{n}} \to 0 \quad \text{and} \quad \frac{\rho_n(r_1, r_3, 0, \tilde{\gamma}_{r_3})}{\rho_n(r_2, r_3, 0, \tilde{\gamma}_{r_3})} \sim \sqrt{\frac{r_1}{r_2}} \to 0, \text{ as } n \to \infty,$$

since, $r_1 =$ constant w.r.t. n and $r_2 \to \infty$, as $n \to \infty$,
and

(III) $\rho_n(r_1, r_2, 0, \tilde{\gamma}_{r_2}), \rho_n(r_2, r_3, 0, \tilde{\gamma}_{r_3}) \to 0$ and $\dfrac{\rho_n(r_2, r_3, 0, \tilde{\gamma}_{r_3})}{\rho_n(r_1, r_2, 0, \tilde{\gamma}_{r_2})} \sim \sqrt{\dfrac{1-\lambda}{\lambda r_1}} \times \dfrac{r_2}{\sqrt{n}} \to 0$, as $n \to \infty$,

if and only if $r_2 = \circ(\sqrt{n})$, as $n \to \infty$, since $r_1 =$ constant w.r.t. n, as $n \to \infty$.

Example 3.1 *(the determination of a suitable type of a given order statistic).* As an interesting application of the above consequence, we consider a requirement of a certain statistical problem which stipulates the asymptotic independence between the two order statistics $X_{10:100}$ and $X_{50:100}$. For performing a goodness of fit test to identify the suitable limit distribution type of each of the statistics $X_{10:100}$ and $X_{50:100}$, we first have to choose their types (extreme or intermediate or central type). The type of the order statistic $X_{50:100}$ can reasonably regarded as a central type, i.e., $X_{50:100} = X^C_{r_3:n}$. However, on the one side, the type of the order statistic $X_{10:100}$ may be regarded as extreme type, i.e., $X_{10:100} = \underline{X}^E_{r_1:n}, r_1 = 10, n = 100$, but on the other side, it may be regarded as lower intermediate type $X_{10:100} = \underline{X}^I_{r_2:n}, r_2 = [\sqrt{100}], n = 100$. In view of our requirement, Corollary 3.3, part (II), enables us to decide that the choice of extreme type for the order statistic $X_{10:100}$ is better than the choice of lower intermediate type.

Example 3.2 *(type II right censored samples)* Let the censoring scheme be $R_1 = R_2 = \cdots = R_{M-1} = 0, R_M = n - M$. Therefore, we get $\beta_j = \gamma_j = 2n + M - j + 1$. Thus, if $r < s$, we get, after simple calculations,

$$\frac{A_s(m)}{A_r(m)} = \frac{A_s(0)}{A_r(0)} = \frac{2n - M - s + 1}{2n - M - r + 1}$$

and

$$\frac{B_r(m) - A_r(m)}{B_s(m) - A_s(m)} = \frac{B_r(0) - A_r(0)}{B_s(0) - A_s(0)} = \frac{r}{s}.$$

The last two relations, thus yield

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{r(2n - M - s + 1)}{s(2n - M - r + 1)}}.$$

Therefore, if M is constant with respect to n then $X^\star(r)$ and $X^\star(s)$ as well as $X^\star_d(r)$ and $X^\star_d(s)$ are dependent for all r, s and n. On the other hand, by assuming that $M = M(n) \to \infty$ and $\frac{M}{n} \to 0$, as $n \to \infty$, we can easily deduce that Theorem 2.1 in [2], which is concerned with the asymptotic dependence between oos, will hold for this model.

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